Geometric Calculation of the Invariant Integral of Classical Groups

Amelia Álvarez Sánchez
Universidad de Extremadura
e-mail: aalarma@unex.es

Carlos Sancho de Salas
Universidad de Salamanca
e-mail: mplu@usal.es

Pedro Sancho de Salas
Universidad de Extremadura
e-mail: sancho@unex.es

Abstract

An affine k-group $G = \operatorname{Spec} A$ is semisimple if and only if A^* splits into the form $A^* = k \times B^*$ as k-algebras, where the first projection $\pi_1\colon A^* \to k$ is the morphism $\pi_1(w) := w(1)$ ([2, Th. 2.6]). The linear form $w_G := (1, 0) \in k \times B^* = A^*$ will be referred to as the *invariant integral* of G.

In the theory of invariants the calculation of the invariant integral w_G is of great interest, because it yields the calculation of the invariants of any representation. The aim of this article is the explicit calculation of w_G when $G = Sl_n, Gl_n$, O_n, Sp_{2n} ($\operatorname{char} k = 0$) by geometric arguments and by means of the Fourier transform, which is defined below. Although G is not a compact group, it is possible to define the invariant integral of G, the Fourier transform, the convolution product and to prove the Parseval identity , the inversion formula, etc.

Let A_i^* be simple (and finite) k-algebras and let $A^* = \prod_i A_i^*$. On every A_i^*, one has the non-singular trace metric and its associated polarity. Hence, one obtains a morphism of A^*-modules $\phi : A = \oplus_i A_i \hookrightarrow \prod_i A_i^* = A^*$. If $G = \operatorname{Spec} A$ is a semisimple affine k-group and $* : A \to A$, $a \mapsto a^*$ is the morphism induced by the morphism $G \to G$, $g \mapsto g^{-1}$, we prove that ϕ is the morphism

$$A \to A^*,\ a \mapsto w_G(a^* \cdot -)$$

where $w_G(a^* \cdot -)(b) := w_G(a^* \cdot b)$. We will call ϕ the Fourier transform. The product operation on A^* defines, via the Fourier transform, a product on A, which is the *convolution product* in the classical examples.

Let us consider a system of coordinates in G, that is, let us consider $G = \operatorname{Spec} A$ as a closed subgroup of a semigroup of matrices $M_n = \operatorname{Spec} B$. Then A is the quotient of B by the ideal I of the functions of M_n vanishing on G. Hence, A^* is a subalgebra of B^* and

one has that $k \cdot w_G = A^{*G} = \{w \in B^{*G} : w(I) = 0\}$. Moreover, B^G (which is the ring of functions of M_n/G), coincides essentially with B^{*G}, via the Fourier transform. Finally, we prove that given $w \in B^{*G}$, the condition $w(I) = 0$ is equivalent to $w(I^G) = 0$, which is a finite system of equations "in each degree".

References

[1] A. Álvarez, C. Sancho, P. Sancho, *Algebra schemes and their representations.* J. Algebra, **Volume**(296) (2006) 110-144.

[2] A. Álvarez, C. Sancho, P. Sancho, *Reynolds operator.* Preprint, arXiv:math/0611311v2 [math.AG] (2006).

[3] M. Brion, G.W. Schwarz, *Théorie des invariants et géometrie des variétés quotients.* Travaux en cours, vol. 61, Hermann, Paris, 2000.

[4] I. Dolgachev, *Lectures on invariant theory.* London Mathematical Society Lecture Notes Series, vol. 296, Cambridge University Press, Cambridge, 2003.

[5] R. Goodman, N.R. Wallach, *Representations and Invariants of the Classical Groups.* Cambridge University Press, New York, 1998.

[6] Ju. B. Ermolaev, *Simultaneous reduction of a pair of bilinear forms to canonical form.* Dokl. Akad. Nauk SSSR, **Volume** (132) 257-259 (Russian); translated as Soviet Math. Dokl. **Volume** (1) (1960) 523-525.

[7] C. Sancho de Salas, *Grupos algebraicos y teoría de invariantes.* Sociedad Matemática Mexicana, México, 2001.