Irreducible numerical semigroups

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Abstract

This is a survey of several results obtained with M. B. Branco at the Universidade de Évora, J. I. García-García at the Universidad de Cádiz, J. A. Jiménez-Madrid at the Instituto de Ciencias Matemáticas, and J. C. Rosales at the Universidad de Granada.

A *numerical semigroup* is a submonoid of the monoid of nonnegative integers (under addition) with finite complement in it. A numerical semigroup is *irreducible* if it cannot be expressed as the intersection of two numerical semigroups properly containing it. Every numerical semigroup can be written as a (finite) intersection of irreducible numerical semigroups. Minimal decompositions can be thought in terms of cardinality or redundancy. Both concepts do not coincide, and uniqueness is far from being reached.

As we have mentioned above, the complement of a numerical semigroup in the set of nonnegative integers is finite. So it makes sense to think about the largest integer not belonging to a numerical semigroup. This integer is known as the *Frobenius number*, which became really famous mainly for the lack of a general (algebraic) formula to calculate it for numerical semigroups with more than two minimal generators. Irreducible numerical semigroups can be also characterized as those numerical semigroups maximal (with respect to set inclusion) in the set of numerical semigroups with a fixed Frobenius number. This allows to relate this concept with two well known families of numerical semigroups: symmetric and pseudo-symmetric numerical semigroups. An irreducible numerical semigroup is either symmetric or pseudo-symmetric (depending on the parity of its Frobenius number), and the union of the set of symmetric and the set of pseudo-symmetric numerical semigroups yields that of irreducible numerical semigroups.

Symmetric numerical semigroups earned some popularity due to a result by Kunz that states that the semigroup ring associated to a numerical semigroup is Gorenstein if and only if the numerical semigroup is symmetric (semigroup rings associated to pseudo-symmetric numerical semigroups are Kunz rings). Several subfamilies of the class of symmetric numerical semigroups were also widely studied due to their relevance as examples in Algebraic Geometry. These include that of complete intersections, free (in the sense of Bertin and Carbonne) and telescopic numerical semigroups.

In the set of numerical semigroups with a given Frobenius number, the average of irreducible numerical semigroups is really poor. However, this apparent lack contrasts not

only with the idea that every numerical semigroup is the intersection of irreducible numerical semigroups, but also with the surprising fact that every numerical semigroup is one half of infinitely many symmetric numerical semigroups. By one half of a numerical semigroup, we mean the set of integers that multiplied by two are in the semigroup. This set is again a numerical semigroup. Analogously, one fourth of a numerical semigroup can be defined. Rosales has recently shown that every numerical semigroup is one fourth of a pseudo-symmetric numerical semigroup.

Another amazing property of irreducible numerical semigroups is that every positive integer can be realized as the Frobenius number of an irreducible numerical semigroup with at most four generators.

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