Universal Banach spaces with a projectional resolution of the identity: category-theoretic approach

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Abstract

A classical result of Pełczyński [1] from 1969 says that there exists a separable Banach space P with a Schauder basis which is complementably universal for all separable Banach spaces with a basis, i.e. every space in this class is isomorphic to a complemented subspace of P. Two years later, Kadec [2] proved a similar result for Banach spaces with the *Banach approximation property* (BAP) – a sequence of projections onto finite-dimensional subspaces converging pointwise to the identity.

A straight generalization of a Schauder basis for a non-separable Banach space is the notion of a *Markushevich basis*, i.e. a bi-orthogonal system $\langle x_\alpha, y_\alpha \rangle_{\alpha\in\Gamma}$ such that $\{x_\alpha\}_{\alpha\in\Gamma}$ is linearly dense and $\{y_\alpha\}_{\alpha\in\Gamma}$ is total. It seems that for Banach spaces of the smallest uncountable density $\aleph_1$, a natural requirement is that the space

$$\{y\colon y(x_\alpha) = 0 \text{ for all but countably many } \alpha\}$$

be norming. Such a basis is called *countably norming*. A Banach space X with a countably norming Markushevich basis has several nice properties. For instance: it admits a bounded 1-1 linear operator into a space of type $c_0(\Gamma)$ and it has an equivalent locally uniformly convex norm on X. The main tool for proving these properties is the notion of a *projectional resolution of the identity* (PRI), that is, a continuous transfinite sequence of pairwise compatible projections onto smaller subspaces, converging pointwise to the identity. An interesting fact is that, for a Banach space of the smallest uncountable density $\aleph_1$, the existence of a PRI is actually equivalent to the existence of a countably norming Markushevich basis. Thus, in the class of spaces of density $\aleph_1$, projectional resolution of the identity is a common generalization of BAP and Schauder basis.

We prove that, assuming the continuum hypothesis $2^{\aleph_0} = \aleph_1$, there exists a Banach space U of density $\aleph_1$ which has a projectional resolution of the identity and which is complementably universal for the class of all spaces with these properties. The space U has moreover some sort of homogeneity property for separable Banach spaces which makes it unique up to a linear isometry.

The space U is constructed by using purely category-theoretic methods. Namely, we only use the following two properties of the category $\mathfrak{B}_{\mathrm{sep}}$ of separable Banach spaces with linear operators of norm $\leqslant 1$.

(1) Every pair of left-invertible arrows in $\mathfrak{B}_{\mathrm{sep}}$ with the same domain admits the pushout in $\mathfrak{B}_{\mathrm{sep}}$.

(2) Every countable inductive sequence consisting of left-invertible arrows in $\mathfrak{B}_{\mathrm{sep}}$ has the colimit in $\mathfrak{B}_{\mathrm{sep}}$.

The talk is based on the work [3].

References

[1] A. Pełczyński, *Universal bases*, Studia Math. 32 (1969) 247–268

[2] M.I. Kadec, *On complementably universal Banach spaces*, Studia Math. 40 (1971) 85–89

[3] W. Kubiś, *Fraïssé sequences: category-theoretic approach to universal homogeneous objects*, preprint, `http://arxiv.org/abs/0711.1683`